Lower Dimension Electric Field is a "Slice" of a Higher Dimension

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Abstract

We have introduced a novel perspective of understanding the electric field generated by a charged object by extending it to higher dimensions. The electric field generated by an arbitrarily charged object can be viewed as if it is "extended" in another direction that is "perpendicular" to the current space. This approach makes some symmetric and beautiful electric behaviours in three dimensions much easier to comprehend.

I. INTRODUCTION

In introductory electrodynamics courses, we are required to calculate the electric field at a point generated by a charged object in a certain shape, usually a uniformly charged straight line or plane or symmetric object like a circle or sphere. Gauss's law[1] or multiple integration[2] are the conventional method for computing them. Although we strengthen our calculation ability by doing so, it seems that the process is repetitive and uninspiring. For example, when we are dealing with the electric field intensity above an infinitely large uniformly charged plane, we get the amazing result that the field is constant regardless of the perpendicular distance between the point and the plane, with a coefficient $1/2\epsilon_0$ at the front. It turns out that only when we look up or down to generalized fields in higher or lower dimensions could we understand the coefficients and behaviours. We proposed a heuristic approach towards this kind of problem.

II. DEFINING GENERALIZED FIELDS

A. Natural and non-natural field of point charges

In order to make sense of "fields" in higher dimension, we first define the two important concepts, natural field and non-natural field in n dimension, in terms of points charges (or "sources"). Let $\tilde{Q}$ be a quantity of the source of the field in $\mathbb{R}^n$, which would be the analogue of a point charge or point mass. (We will stick to using "point charge" instead of the more general term "source" to avoid confusion. Furthermore, we add a tilde over the symbols or use subscript n to emphasize that it is the analogue quantity in dimension n) Intuitively, if the field generated by $\tilde{Q}$ decays at the rate of $1/r^{n-1}$, we say the field is *natural* in $\mathbb{R}^n$. If not, it is called *non-natural* in $\mathbb{R}^n$, i.e.,

Definition. *Let $\tilde{Q}$ a generalized point charge at the origin in $\mathbb{R}^n$. If the field intensity follows the form*

$$\mathbf{E}_n(\boldsymbol{r}) = \tilde{k}_n \frac{\tilde{Q}}{r^{n-1}} \hat{\boldsymbol{r}}, \tag{1}$$

where $\tilde{k}_n$ is a constant coefficient (fixed in each dimension). We say the field generated by the point charge is natural in $\mathbb{R}^n$.

When $n = 3$, many authors use the term "coulombic" or "inverse-square". We mainly deal with natural fields in this article. Now we should figure out the constants $\tilde{k}_n$ in different dimensions. We know in $\mathbb{R}^3$ that $\tilde{k}_3 = 1/(4\pi\epsilon_0)$. It *can* be considered as a special case of Gauss's law, where we take the Gaussian surface to be the unit sphere centred at the origin. Since the surface area of the unit sphere is 4π, we have that number in the denominator. We assume the form of Guass's law remains unchanged in higher dimension (ref needed), i.e., Equation (2) is set to be applied to all dimensions, where $\tilde{\epsilon_0}$ is the hypothetical vacuum permittivity in $\mathbb{R}^n$ (ref needed).

$$\oiint_S \mathbf{E}_n \cdot \mathrm{d}\boldsymbol{A} = \frac{\tilde{Q}_{\mathrm{in}}}{\tilde{\epsilon_0}} \tag{2}$$

If we combine equation (1) and (2), we have the equation (3) for $\tilde{k}_n$ in $\mathbb{R}^n$, we can see that the coefficient $\tilde{k}_n$ actually has different value in different dimensions:

$$\tilde{k}_n = \frac{1}{A_n \tilde{\epsilon_0}}, \tag{3}$$

where A_n represents the surface area[3] of the unit sphere S embedded in $\mathbb{R}^n$ given by equation (4). It is worth noting that Guass's law in the form of equation (2) only holds when the field is natural. For non-natural fields, since the decay rate of surface area A_n $(1/r^{n-1})$ is not equal to that of the electric field, the flux of the field through a surface S may change as the shape of S varies. (ref needed)

$$A_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \begin{cases} \frac{2k}{k!}\pi^k & n = 2k, \\ \frac{(2k+1)2^{k+1}}{(2k+1)!!}\pi^k & n = 2k+1. \end{cases} \tag{4}$$

B. Examples

Example 1. TABLE I lists the surface area of the unit sphere and the expression for natural fields in some dimension. We pay special attention to the case of $n = 1$ and $n = 2$ because we will show later that this is exactly what happens to the field generated by an infinite plane and line.

Example 2. Show that the natural field within a hypersphere radius R in $\mathbb{R}^n$ is zero.

Solution. This claim become obvious once you conceive that Gauss's law applies to natural fields in arbitrary dimensions. (Take a concentric sphere of radius less than R as the Gaussian surface).

TABLE I. Relation between surface area A_n and the natural field of point charges in $\mathbb{R}^n$

Dimension n	1	2	3	4	5	6	$\cdots$	n
Surface area of unit sphere A_n	2	2π	4π	$2\pi^2$	$\frac{8\pi^2}{3}$	π^3	$\cdots$	$\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$
Natural field expression $\mathbf{E}_n$	$\frac{\tilde{Q}}{2\tilde{\epsilon_0}}\hat{\boldsymbol{r}}$	$\frac{\tilde{Q}}{2\pi\tilde{\epsilon_0}r}\hat{\boldsymbol{r}}$	$\frac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$	$\frac{\tilde{Q}}{2\pi^2\tilde{\epsilon_0}r^3}\hat{\boldsymbol{r}}$	$\frac{3\tilde{Q}}{8\pi^2\tilde{\epsilon_0}r^4}\hat{\boldsymbol{r}}$	$\frac{\tilde{Q}}{\pi^3\tilde{\epsilon_0}r^5}\hat{\boldsymbol{r}}$	$\cdots$	$\frac{\Gamma(\frac{n}{2})\tilde{Q}}{2\pi^{\frac{n}{2}}\tilde{\epsilon_0}r^{n-1}}\hat{\boldsymbol{r}}$

III. EXTENDING A POINT CHARGE TO ANOTHER DIMENSION

Consider the following case: Let a point charge $\tilde{Q}$ sitting at the origin in $\mathbb{R}^n$ and the natural field generated by the charge at point $\boldsymbol{r}$ is $\tilde{k}_n\tilde{Q}\hat{\boldsymbol{r}}/r^{n-1}$. Imagine extending that point charge to a line (one-dimensional) in $\mathbb{R}^{n+1}$ having a uniform line charge density $\tilde{\lambda}$. The natural field generated by the charged line at a point distance $\boldsymbol{d}$ from the line is $\mathbf{E}_{n+1}(\boldsymbol{d})$. We claim that $\mathbf{E}_n(\boldsymbol{r})$ and $\mathbf{E}_{n+1}(\boldsymbol{d})$ have exactly the same form, explicitly,

Claim.

$$\mathbf{E}_{n+1}(\boldsymbol{d}) = \tilde{k}_n\frac{\tilde{\lambda}}{d^{n-1}}\hat{\boldsymbol{d}}. \tag{5}$$

Proof. Since it would be hard to define and find a surface with some part of the charged line enclosed in an abstract high-dimensional space, we use the integral method. Remember $\mathbb{R}^n$ is also an inner product space, it is legal to use the concept of "angle", etc.

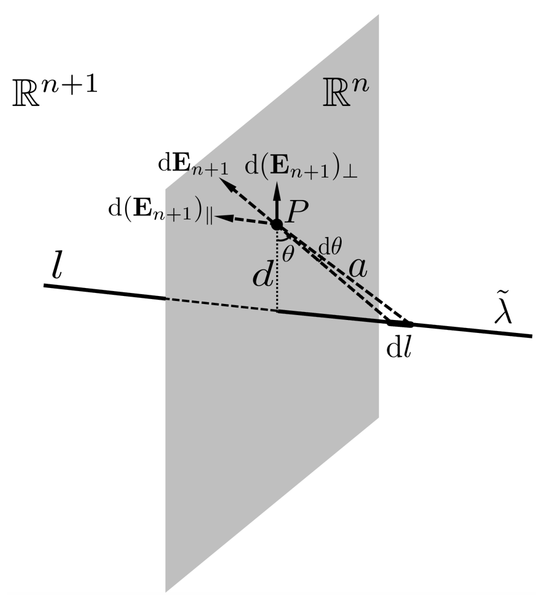

FIG. 1. Proof of the claim by integrating infinitesimal line segments in another dimension

Consider the situation in Fig. 1. Suppose point P is at distance d to the charged line l, let $\mathrm{d}l$ be a differential segment on the line. The electric field generated by $\mathrm{d}l$ at P is

$$|\mathrm{d}\mathbf{E}_{n+1}| = \tilde{k}_{n+1}\frac{\tilde{\lambda}\mathrm{d}l}{a^n}. \tag{6}$$

By symmetry, the parallel component $\mathrm{d}(\mathbf{E}_{n+1})_{\parallel}$ will be cancelled out after integration, what we care about is the vertical component $|\mathrm{d}(\mathbf{E}_{n+1})_{\perp}|$. By substituting $\mathrm{d}l = a\mathrm{d}\theta/\cos\theta$ and $a = d/\cos\theta$, we get:

$$\begin{aligned}
|\mathrm{d}(\mathbf{E}_{n+1})_{\perp}| &= |\mathrm{d}\mathbf{E}_{n+1}|\cos\theta = \tilde{k}_{n+1}\frac{\tilde{\lambda}\mathrm{d}l}{a^n}\cos\theta \\
&= \tilde{k}_{n+1}\frac{\tilde{\lambda}(\frac{d}{\cos\theta})\mathrm{d}\theta\cdot\frac{1}{\cos\theta}}{(\frac{d}{\cos\theta})^n}\cos\theta \\
&= \tilde{k}_{n+1}\frac{\tilde{\lambda}}{d^{n-1}}(\cos\theta)^{n-1}\mathrm{d}\theta,
\end{aligned} \tag{7}$$

with θ ranging from $-\pi/2$ to $\pi/2$. The field at point P is given by the following integral:

$$|\mathbf{E}_{n+1}(\boldsymbol{d})| = \tilde{k}_{n+1}\frac{\tilde{\lambda}}{d^{n-1}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\theta)^{n-1}\mathrm{d}\theta. \tag{8}$$

The integral on the RHS of equation (8) is a famous integral known as the Wallis integral W_{n-1} (Some authors[4] use S_n for Wallis integral). Its value is shown below:

$$W_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\theta)^n\mathrm{d}\theta = \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2}+1)} = \begin{cases}\pi\cdot\frac{(2k-1)!!}{(2k)!!} & n = 2k,\\ 2\cdot\frac{(2k)!!}{(2k+1)!!} & n = 2k+1.\end{cases} \tag{9}$$

Compare equation (5) and (8), our mission is to proof that $\tilde{k}_{n+1}\cdot W_{n-1} = \tilde{k}_n$. Using the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have:

$$\begin{aligned}
\tilde{k}_{n+1}\cdot W_{n-1} &= \frac{1}{A_{n+1}\tilde{\epsilon_0}}\cdot W_{n-1} \\
&= \frac{1}{\left(\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\right)\tilde{\epsilon_0}}\cdot\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})} \\
&= \frac{\Gamma(\frac{n+1}{2})}{2\pi^{\frac{n}{2}}\sqrt{\pi}\tilde{\epsilon_0}}\cdot\frac{\Gamma(\frac{n}{2})\sqrt{\pi}}{\Gamma(\frac{n+1}{2})} \\
&= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}\tilde{\epsilon_0}} = \tilde{k}_n.
\end{aligned} \tag{10}$$

□

This surprising result can be interpreted in the following way: Imagine some creatures living in n dimensional space (say $n = 3$) measuring the field (suppose it is *natural*) generated by a point charge $\tilde{Q}$ at the origin. The creatures would not feel any difference if the point is actually an infinite line with uniform charge density $\tilde{\lambda}$ (equals $\tilde{Q}$ numerically) in $n + 1$ dimensional space (4-dimensional in this case).

Furthermore, since the vector field can be added together linearly (superposition principle), any shape with some charges on it can be viewed as if every charge on it were extended in another dimension, generating fields in that higher dimension, and still feel the same in a cross-section (or a perpendicular "slice" of that space)!

IV. DEMONSTRATION

We will use the conclusion in III to show two examples in electrostatics. It appears that the process is elegant and neat. Note that the following examples are in our familiar $\mathbb{R}^3$.

Example 3. Show that the electric field generated by an infinitely large uniformly charged plane (with surface charge density σ) and an infinitely long line (with line charge density λ) at point P distance d to the object are $\sigma\hat{\boldsymbol{d}}/(2\epsilon_0)$ and $\lambda\hat{\boldsymbol{d}}/(2\pi\epsilon_0 d)$, respectively.

Solution. By the reverse logic of the claim in III, as shown in Fig. 2, every infinite line l on the plane can be shrunk down to a point K. After the squeeze process, the plane becomes an infinite line with charge density $\tilde{\lambda}$. Then, we should reconsider the problem in a lower dimension $\mathbb{R}^2$. Again, that line can be compressed to a point with charge $\tilde{Q}$ in an even lower dimension $\mathbb{R}^1$. Now we simplify the original problem to a simple problem, that is to find the natural field generated by a point charge in $\mathbb{R}^1$, which is obviously $\mathbf{E}_1 = \tilde{Q}\hat{\boldsymbol{d}}/(2\tilde{\epsilon_0})$ (c.f. Example 1). Therefore, the original field should have a similar form: $\sigma\hat{\boldsymbol{d}}/(2\epsilon_0)$.

By the exact same logic, we can compress an infinite line in $\mathbb{R}^3$ to a point in $\mathbb{R}^2$. Since the natural field generated by $\tilde{Q}$ in $\mathbb{R}^2$ is $\mathbf{E}_2 = \tilde{Q}\hat{\boldsymbol{d}}/(2\pi\tilde{\epsilon_0}r)$ (c.f. Example 1), the electric field at P distance d to the line l'' is $\lambda\hat{\boldsymbol{d}}/(2\pi\epsilon_0 d)$.

Example 4. Show that the electric field inside of an infinitely long uniformly charged cylinder shell is zero.

Solution. Let's first consider a more general case where the cross-section of the cylinder is some arbitrary curve C shown in Fig. 3. Every strip l on the shell can be decomposed into

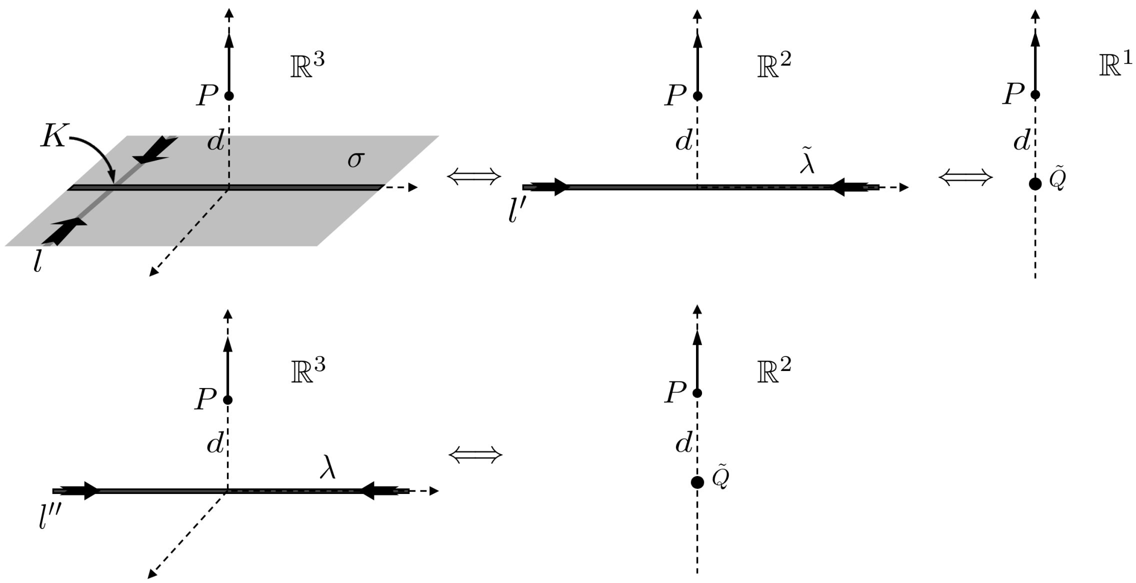

FIG. 2. Lower dimensional analogue of the field generated by a plane and a line in $\mathbb{R}^3$

a point K. It turns out we are allowed to only consider the cross-section of the cylinder in a lower dimension $\mathbb{R}^2$. If the curve C is a perfect circle, by Example 2, the field inside C is zero everywhere. Therefore, the field inside the cylinder must be zero.

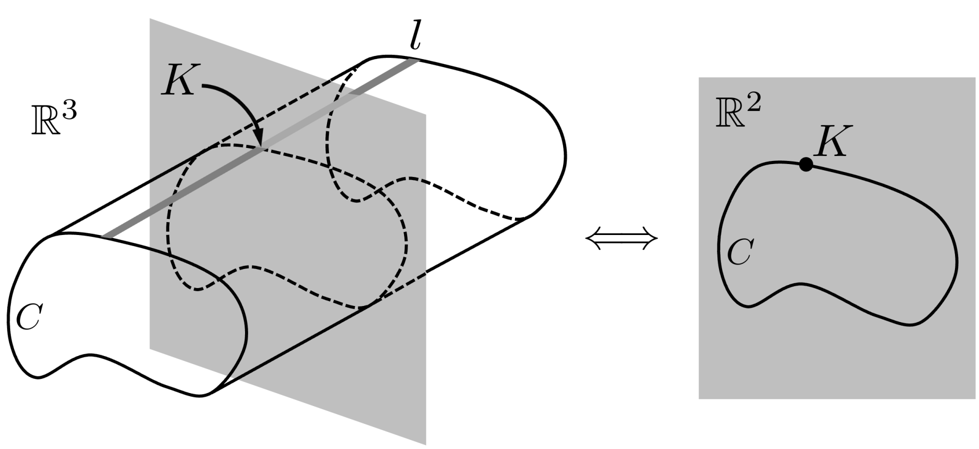

FIG. 3. The field inside a cylinder can be simplified to the field inside a curve

V. CONCLUSION

The field generated by a charged object can be viewed as if the object itself is "linearly extended" in another dimension and generating *natural* fields in that higher dimensional space. Conversely, the field from a linear-ish object can be simplified to a problem in a lower dimension, which is a subspace (or a cross-section, a "slice") of the higher dimension. This amazing result makes many problems much easier to solve and understand.

ACKNOWLEDGMENTS

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